University of Groningen

Examination - 2024/25

Date : 09/04/2025		Total number of points: 90
Final Examination	Relativistic Quantum Mechanics	Final grade = 1 + (points)/10

<u>Follow these instructions</u>:

- Your answers should be legible (if it is not readable, it will not be marked at all).

- You can follow David Tong's clean printed lecture notes (without your handwritten additions / derivations).

- There are three exercises, each consisting of three subquestions. All subquestions are worth 10 points.

Exercise 1: Classical field theory

(a) Consider a scalar field with Lagrangian density given by

$$\mathcal{L} = \tfrac{1}{2}(\partial\phi)^2 - \tfrac{1}{2}m^2\phi^2 - \tfrac{1}{24}\lambda\phi^4\,, \tag{1}$$

which consists of a massive real scalar field with a quartic interaction (given by the last term). What is the Euler-Lagrange equation for this Lagrangian?

(b) What is the energy-momentum tensor $T_{\mu\nu}$ corresponding to the above Lagrangian?

(c) What is the mass dimension of the different symbols in the above Lagrangian, i.e. ∂, ϕ, m and λ?

Exercise 2: Quantizing a real scalar field

(a) Consider a free scalar field ϕ with Hamiltonian

$$H = \int d^3x(\tfrac{1}{2}(\dot{\phi})^2 + \tfrac{1}{2}(\vec{\nabla}\phi)^2 + \tfrac{1}{2}m^2\phi^2)\,. \tag{2}$$

In canonical quantization, the field ϕ and its conjugate momentum $\pi = \dot{\phi}$ are replaced by operators, subject to the quantum commutation relation

$$[\phi(\vec{x}), \pi(\vec{y})] = i\delta^{(3)}(\vec{x} - \vec{y})\,. \tag{3}$$

Calculate $[H, \phi(\vec{x})]$.

(b) It is often convenient to express the field and momentum operators in terms of ladder operators, which are then subject to quantum commutation relations

$$[a_{\vec{p}}, a^{\dagger}_{\vec{q}}] = (2\pi)^3 \delta^{(3)}(\vec{p} - \vec{q})\,. \tag{4}$$

Write the following product in normal order (that is, write as a sum of terms with the order $a^{\dagger} a^{\dagger} \cdots a a$ only):

$$a_{\vec{p}_1} a^{\dagger}_{\vec{p}_2} a^{\dagger}_{\vec{p}_3}\,. \tag{5}$$

(c) One can construct a one-particle state of the form $\phi(x)|0>$ (in the Heisenberg picture). The inner product between two such states is $D(x-y) \equiv < 0|\phi(x)\phi(y)|0 >$. When calculated at equal times, it decays like

$$D(x - y) \sim e^{-m|\vec{x}-\vec{y}|}\,, \tag{6}$$

as a function of the spatial separation of the two points x and y. When calculated for two points that are not simultaneous and still outside of each other's lightcone, does this inner product strictly vanish or decay exponentially? Clearly state your answer (vanishes / decays) and briefly explain your argument for it (in one or two sentences).

Exercise 3: Dirac equation of a fermion field

(a) Consider a free fermion field $\psi(x)$ with energy-momentum tensor

$$T_{\mu\nu} = i\bar{\psi}\gamma_{\mu}\partial_{\nu}\psi\,. \tag{7}$$

Prove that this tensor is conserved, i.e. $\partial_{\mu}T^{\mu\nu} = 0$, when imposing the Dirac equation.

(b) The Hamiltonian in terms of ladder operators is given by

$$H = \sum_{s=1,2} \int \frac{d^3p}{(2\pi)^3} E_{\vec{p}} (b^{s\dagger}_{\vec{p}} b^{s}_{\vec{p}} + c^{s\dagger}_{\vec{p}} c^{s}_{\vec{p}})\,. \tag{8}$$

One can construct an eigenstate of the Hamiltonian by acting with the creation operators on the vacuum. What is the eigenvalue (i.e. the energy) of the two-particle state $b^{s_1\,\dagger}_{\vec{p}_1} c^{s_2\,\dagger}_{\vec{p}_2}|0>$?

(c) The Dirac equation builds on the Clifford algebra for γ-matrices, defines as $\{\gamma^{\mu}, \gamma^{\nu}\} = 2\eta^{\mu\nu}\mathbb{I}_4$. From these, one can define the spinorial representation of Lorentz generators as $S^{\mu\nu} \equiv \frac{1}{4}[\gamma^{\mu}, \gamma^{\nu}]$. The commutators of S with either a single γ-matrix, $[S^{\mu\nu}, \gamma^{\rho}]$, or with itself, $[S^{\mu\nu}, S^{\rho\sigma}]$, are calculated explicitly in Tong. Instead, calculate the commutator of S with the anti-commutator of two γ-matrices:

$$[S^{\mu\nu}, \{\gamma^{\rho}, \gamma^{\sigma}\}]\,. \tag{9}$$